4. Is it possible for an infinite group to have a finite subgroup?

5. Let $G$ be a finite group and $H$ a subgroup of $G$. What is $H$ if

   (i) $[G : H] = 1$ (ii) $[G : H] = |G|$.

6. Determine which of the following statements are true and which are false.

   (a) The index of any subgroup of an infinite group is infinite.

   (b) Any group having no proper subgroup is cyclic.

   (c) Every group of prime order is abelian.

   (d) Every group of prime order is cyclic.

   (e) A subgroup of a group is a left coset of itself.

   (f) In a finite group the order of any element divides the order of the group.

   (g) For any subgroup of a group, the number of left cosets is the same as the number of right cosets.

   (h) $\mathbf{Z}_{12}$ contains a subgroup of order 3.

   (i) Any infinite group has proper subgroups.

   (j) Any group which is not cyclic has proper subgroups.

**Answers.**

5. (a) $H = G$. (b) $H = \{e\}$

6. (a) is false and the other statements are true.

## 3.9. Normal Subgroups and Quotient Groups

Consider the subgroup $H = \{e, p_3\}$ of $S_3$. Then $Hp_1 = \{p_1, p_5\}$ and $p_1H = \{p_1, p_4\}$. Hence $Hp_1 \neq p_1H$. Thus a left coset need not be equal to the corresponding right coset. However there are subgroups for which every left coset is the same as the corresponding right coset.

For example, consider the subgroup

$H = \{e, p_1, p_2\}$ of $S_3$.

Clearly $Ha = aH = H = \{e, p_1, p_2\}$ for all $a \in H$ and $aH = Ha = S_3 - H = \{p_3, p_4, p_5\}$ for all $a \notin H$.

$\therefore \quad aH = Ha$ for all $a \in S_3$.

Thus for some subgroups every left coset is a right coset. This leads to the following definition of a special class of subgroups.

**Definition.** A subgroup $H$ of $G$ is called **a normal subgroup** of $G$ if $aH = Ha$ all $a \in G$.

**Examples**

1. For any group $G$, $\{e\}$ and $G$ are normal subgroups.

2. In $S_3$, the subgroup $\{e, p_1, p_2\}$ is normal.

3. In $S_3$, the subgroup $\{e, p_3\}$ is not a normal subgroup.

**Theorem 3.39.** Every subgroup of an abelian group is a normal subgroup.

**Proof.** Let $G$ be an abelian group and let $H$ be a subgroup of $G$. Let $a \in G$.

We claim that $aH = Ha$.

Let $x \in aH$. Then

$$x = ah \text{ for some } h \in H.$$
$$= ha \text{ (since } G \text{ is abelian).}$$

$\therefore \quad x \in Ha$. Hence $aH \subseteq Ha$.

Similarly $Ha \subseteq aH$.

$\therefore \quad aH = Ha$ and hence $H$ is a normal subgroup of $G$.

**Examples**

(i) $n\mathbf{Z}$ is a normal subgroup of $(\mathbf{Z}, +)$.

(ii) Every subgroup of $(\mathbf{Z}_n, \oplus)$ is normal.

(iii) Since any cyclic group is abelian any subgroup of a cyclic group is normal.

**Theorem 3.40.** Let $H$ be a subgroup of index 2 in a group $G$. Then $H$ is a normal subgroup of $G$.

**Proof.** If $a \in H$ then $H = aH = Ha$.

If $a \notin H$, then $aH$ is a left coset different from $H$.

Hence $H \cap aH = \Phi$.

Further, since index of $H$ in $G$ is 2, $H \cup aH = G$.

Hence $aH = G - H$.

Similarly $Ha = G - H$ so that $aH = Ha$.

Hence $H$ is a normal subgroup of $G$.

**Example.** The alternating group $A_n$ is a subgroup of index 2 in $S_n$ and hence is a normal subgroup of $S_n$.

**Theorem 3.41.** Let $N$ be a subgroup of $G$. Then the following are equivalent.

(i) $N$ is a normal subgroup of $G$.

(ii) $aNa^{-1} = N$ for all $a \in G$.

(iii) $aNa^{-1} \subseteq N$ for all $a \in G$.

(iv) $ana^{-1} \in N$ for all $n \in N$ and $a \in G$.

**Proof.** (i) $\Rightarrow$ (ii)

Suppose $N$ is a normal subgroup of $G$.

$\therefore \quad aN = Na$ for all $a \in G$.

$\therefore \quad aNa^{-1} = Naa^{-1} = Ne = N$.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i).

Suppose that $ana^{-1} \in N$ for all $n \in N$ and $a \in G$. We claim that $aN = Na$.

Let $x \in aN$.

$\therefore \quad x = an$ for some $n \in N$.

$\therefore \quad x = (ana^{-1})a \in Na$(since $ana^{-1} \in N$).

$\therefore \quad aN \subseteq Na. \quad \ldots (1)$

Now, let $x \in Na$.

$\therefore \quad x = na$ for some $n \in N$.

$\therefore \quad x = a(a^{-1}na) = a(a^{-1}n(a^{-1})^{-1}) \in aN$.

$\therefore \quad Na \subseteq aN. \quad \ldots (2)$

From (1) and (2) we get $Na = aN$.

Hence $N$ is a normal subgroup of $G$.

## Solved problems

**Problem 1.** Prove that the intersection of two normal subgroups of a group $G$ is a normal subgroup of $G$.

**Solution.** Let $H$ and $K$ be two normal subgroups of $G$. Then $H \cap K$ is a subgroup of $G$.

Now, let $a \in G$ and $x \in H \cap K$. Then $x \in H$ and $x \in K$.

Since $H$ and $K$ are normal $axa^{-1} \in H$ and $axa^{-1} \in K$.

Hence $axa^{-1} \in H \cap K$. Thus $H \cap K$ is a normal subgroup of $G$.

**Problem 2.** The centre $H$ of a group $G$ is a normal subgroup of $G$.

**Solution.** The centre $H$ of $G$ is given by

$$H = \{a / a \in G, ax = xa \text{ for all } x \in G\}.$$

Now let $x \in H$ and $a \in G$. Hence $ax = xa$.

$\therefore \quad x = axa^{-1} \in H$.

Hence $H$ is a normal subgroup of $G$.

**Problem 3.** Let $H$ be a subgroup of $G$. Let $a \in G$. Then $aHa^{-1}$ is a subgroup of $G$.

**Solution.** $e = aea^{-1} \in aHa^{-1}$ and hence $aHa^{-1} \neq \Phi$.

Now, let $x, y \in aHa^{-1}$.

Then $x = ah_1a^{-1}$ and $y = ah_2a^{-1}$ where $h_1, h_2 \in H$.

Now,

$$\begin{aligned} xy^{-1} &= (ah_1a^{-1})(ah_2a^{-1})^{-1} \\ &= (ah_1a^{-1})(ah_2^{-1}a^{-1}) \\ &= a(h_1h_2^{-1})a^{-1} \in aHa^{-1} \end{aligned}$$

$\therefore \quad aHa^{-1}$ is a subgroup of $G$.

**Problem 4.** Show that if a group $G$ has exactly one subgroup $H$ of given order, then $H$ is a normal subgroup of $G$.

**Solution.** Let the order of $H$ be $m$.

Let $a \in G$. Then by solved problem 3, $aHa^{-1}$ is also a subgroup of $G$.

We claim that $|H| = |aHa^{-1}| = m$.

Now, consider $f : H \to aHa^{-1}$ defined by $f(h) = aha^{-1}$.

$f$ is $1-1$, for,

$$f(h_1) = f(h_2) \Rightarrow ah_1a^{-1} = ah_2a^{-1}$$
$$\Rightarrow h_1 = h_2.$$

$f$ is onto, for, let $x = aha^{-1} \in aHa^{-1}$.

Then $f(h) = x$. Thus $f$ is a bijection.

$\therefore$ $|H| = |aHa^{-1}| = m$.

But $H$ is the only subgroup of $G$ of order $m$.

$\therefore$ $aHa^{-1} = H$. Hence $aH = Ha$.

$\therefore$ $H$ is a normal subgroup of $G$.

**Problem 5.** Show that if $H$ and $N$ are subgroups of a group $G$ and $N$ is normal in $G$, then $H \cap N$ is normal in $H$. Show by an example that $H \cap N$ need not be normal in $G$.

**Solution.** Let $x \in H \cap N$ and $a \in H$.

We claim that $axa^{-1} \in H \cap N$.

Now, $x \in N$ and $a \in H \Rightarrow axa^{-1} \in N$ (since $N$ is a normal subgroup).

Also $x \in H$ and $a \in H \Rightarrow axa^{-1} \in H$ (since $H$ is a group).

Hence $axa^{-1} \in H \cap N$.

$\therefore$ $H \cap N$ is a normal subgroup of $H$.

The following example shows that $H \cap N$ need not be normal in $G$.

Let $G = S_3$. Take $N = G$ and $H = \{e, p_3\}$.

Now $H \cap N = H$ which is not normal in $G$.

**Problem 6.** If $H$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$ then $HN$ is a subgroup of $G$.

**Solution.** To prove that $HN$ is a subgroup of $G$, it is enough if we prove that $HN = NH$ (Theorem 3.21)

Let $x \in HN$. Then $x = hn$ where $h \in H$ and $n \in N$.

$\therefore$ $x \in hN$.

But $hN = Nh$ (since $N$ is normal).

$\therefore$ $x \in Nh$. Hence $x = n_1h$ where $n_1 \in N$.

$\therefore$ $x \in NH$. Hence $HN \subseteq NH$.

Similarly $NH \subseteq HN$.

$\therefore$ $HN = NH$. Hence $HN$ is a subgroup of $G$.

**Problem 7.** $M$ and $N$ are normal subgroups of a group $G$ such that $M \cap N = \{e\}$. Show that every element of $M$ commutes with every element of $N$.

**Solution.** Let $a \in M$ and $b \in N$.

We claim that $ab = ba$.

Consider the element $aba^{-1}b^{-1}$.

Since $a^{-1} \in M$ and $M$ is normal, $ba^{-1}b^{-1} \in M$.

Also $a \in M$, so that $aba^{-1}b^{-1} \in M$.

Again, since $b \in N$ and $N$ is normal, $aba^{-1} \in N$.

Also $b^{-1} \in N$, so that $aba^{-1}b^{-1} \in N$.

Thus $aba^{-1}b^{-1} \in M \cap N = \{e\}$.

$\therefore$ $aba^{-1}b^{-1} = e$, so that $ab = ba$.

**Exercises.**

1. If $A$ is normal in $G$ and $B$ a subgroup of $G$ such that $A \subseteq B \subseteq G$, then prove that $A$ is a normal subgroup of $B$.
2. Show that a subgroup of index 3 need not be normal.
3. Let $A, B, C, D$ be subgroups of $G$. Let $A$ be a normal subgroup of $B$, and $C$ a normal subgroup of $D$. Then show that $A \cap C$ is a normal subgroup of $B \cap D$.
4. Let $G, G'$ be two groups with identity $e$ and $e'$ respectively. Prove that the subsets $G \times \{e'\}$ and $\{e\} \times G'$ are normal subgroups of $G \times G'$.
5. Let $H$ and $N$ be normal subgroups of a group $G$. Prove that $HN$ is a normal subgroup of $G$.

Let $G$ be a group and $N$ be a subgroup of $G$. We denote by $G/N$ the set of all right cosets of $N$ in $G$. Thus $G/N = \{Na / a \in G\}$. The following theorem enables us to introduce a group structure on $G/N$ when $N$ is a normal subgroup of $G$.

**Theorem 3.42.** A subgroup $N$ of $G$ is normal iff the product of two right cosets of $N$ is again a right coset of $N$.

**Proof.** Suppose $N$ is a normal subgroup of $G$.

Then

$$\begin{aligned} NaNb &= N(aN)b \\ &= N(Nab) \text{ (since } aN = Na) \\ &= NNab \\ &= Nab \text{ (since } NN = N). \end{aligned}$$

Conversely suppose that the product of any two right cosets of $N$ is again a right coset of $N$. Then $NaNb$ is a right coset of $N$.

Further $ab = (ea)(eb) \in NaNb$.

Hence $NaNb$ is the right coset containing $ab$.

$\therefore \quad NaNb = Nab$.

Now, we prove that $N$ is a normal subgroup of $G$.

Let $a \in G$ and $n \in N$. Then

$ana^{-1} = eana^{-1} \in NaNa^{-1} = Naa^{-1} = N$.

$\therefore \quad ana^{-1} \in N$.

Hence $N$ is a normal subgroup of $G$.

**Theorem 3.43.** Let $N$ be a normal subgroup of a group $G$. Then $G/N$ is a group under the operation defined by $NaNb = Nab$.

**Proof.** By theorem 3.42 the operation given by $NaNb = Nab$ is a well defined binary operation in $G/N$.

Now, let $Na, Nb, Nc \in G/N$.

$$\text{Then} \quad \begin{aligned} Na(NbNc) &= Na(Nbc) \\ &= Na(bc) \\ &= N(ab)c \\ &= (NaNb)Nc. \end{aligned}$$

$\therefore$ The binary operation is associative.

$Ne = N \in G/N$ and

$NaNe = Nae = Na = NeNa$.

$\therefore$ $Ne$ is the identity element.

Also $NaNa^{-1} = Naa^{-1} = Ne = Na^{-1}Na$.

$\therefore \quad Na^{-1}$ is the inverse of $Na$.

$\therefore \quad G/N$ is a group.

**Definition.** Let $N$ be a normal subgroup of $G$. Then the group $G/N$ is called the **quotient group (factor group) of $G$ modulo $N$.**

**Example.** $3\mathbf{Z}$ is a normal subgroup of $(\mathbf{Z}, +)$. The quotient group $\mathbf{Z}/3\mathbf{Z} = \{3\mathbf{Z}+0, 3\mathbf{Z}+1, 3\mathbf{Z}+2\}$. Hence $Z/3Z$ is a group of order 3.

**Exercises**

1. Find the order of the following quotient groups.

   (a) $\mathbf{Z}_6/\langle 3\rangle$. (b) $\mathbf{Z}_{60}/\langle 5\rangle$.

2. Compute $S_n/A_n$.

3. Compute $V_4/\{e, a\}$.

4. Prove that if $G$ is abelian and $H$ is a subgroup of $G$, then $G/H$ is abelian.

5. Prove that if $G$ is a cyclic group and $H$ is a subgroup of $G$, then $G/H$ is cyclic.

6. Show that $|G/H| = \dfrac{|G|}{|H|}$ if $H$ is a normal subgroup of a finite group $G$.

7. Determine which of the following statements are true and which are false.

   (a) $\mathbf{Z}/n\mathbf{Z}$ is cyclic.

   (b) Quotient group of any cyclic group is cyclic.

   (c) Quotient group of an abelian group is abelian.

   (d) $S_n/A_n$ is abelian.

   (e) The quotient group of a non-abelian group is non-abelian.

**Answers.**

1. (a) 3 (b) 5
7. (a) T (b) T (c) T (d) T (e) F.